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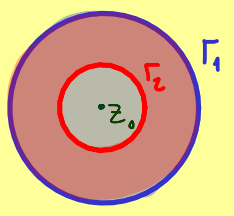
ON STEADY

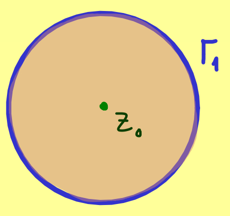
The Bright Side of Mathematics

Complex Analysis – Part 16

Laurent series: $\sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k$ with domain



If the principal part is finite:



← pole at z_0

Definition: Let $f: U \to \mathbb{C}$ be given by a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k$.

If there is $N \in \{-1,-2,\dots\}$ such that $a_k = 0$ for all $k < N$

and $a_N \neq 0$, then we say f has a pole of order $|N|$ at z_0.

Example: (a) $f(z) = \frac{1}{z}$ ← Laurent series $\sum_{k=-\infty}^{\infty} a_k \cdot z^k$

$\Rightarrow f$ has a pole of order 1 at 0.

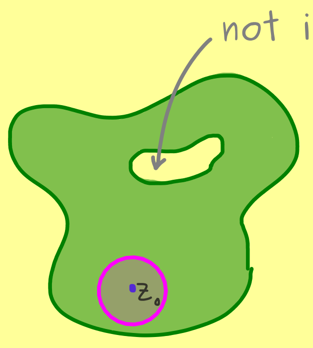
(b) $f(z) = \frac{1}{z} + \frac{1}{z^2} \Rightarrow f$ has a pole of order 2 at 0.

Definition: Let $f: U \to \mathbb{C}$ be holomorphic and $z_0 \notin U$.

If there is $\varepsilon > 0$ with $B_\varepsilon(z_0) \setminus \{z_0\} \subseteq U$,

then z_0 is called an isolated singularity of f.

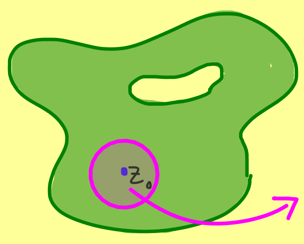
not isolated



Example: $f(z) = \frac{1}{z(z-1)}$ is holomorphic with domain $\mathbb{C} \setminus \{0, 1\}$

isolated singularities

Proposition: At isolated singularities, we always find a Laurent series locally:



proof later

$$B_\varepsilon(z_0) \setminus \{z_0\} \ni z \longmapsto \sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k = f(z)$$

uniquely given

Three cases for isolated singularities:

(1) removable singularity: $\forall k < 0 : \; a_k = 0$

(2) pole: $\exists N \in \{-1,-2,\dots\} \; \forall k < N : \; a_k = 0$ and $a_N \neq 0$

(3) essential singularity: $\forall N \in \{-1,-2,\dots\} \; \exists k \leq N \; a_k \neq 0$

Examples: (1) $f(z) = \frac{\sin(z)}{z} = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!}$ $z_0 = 0$ removable singularity

(2) $f(z) = \frac{\sin(z)}{z^2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k+1)!}$ $z_0 = 0$ pole of order 1

(3) $f(z) = \exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$ $z_0 = 0$ essential singularity