The Bright Side of Mathematics

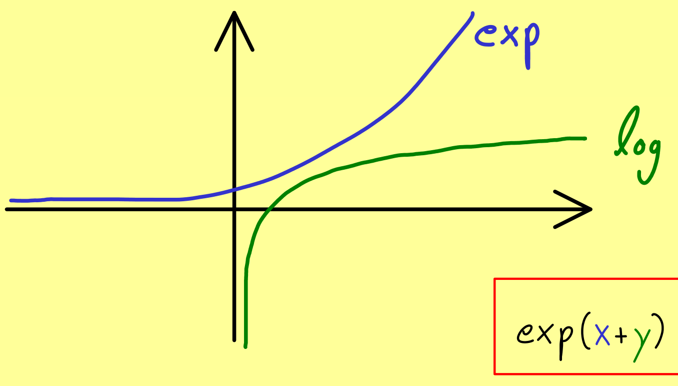
Complex Analysis – Part 13

logarithm $\log$ = inverse function of $\exp$

In $\mathbb{R}$:

$\exp: \mathbb{R} \longrightarrow (0,\infty)$

$\log: (0,\infty) \longrightarrow \mathbb{R}$



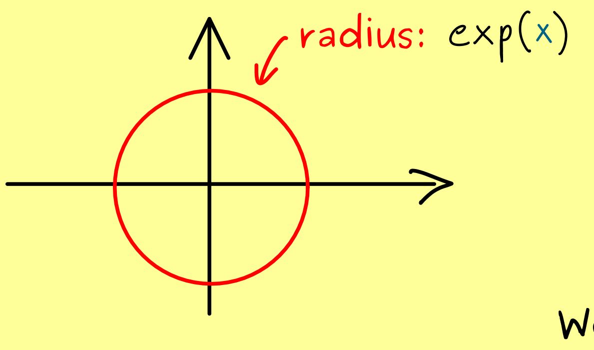
$\boxed{\exp(x+y) = \exp(x)\cdot\exp(y)}$

In $\mathbb{C}$: $z = x + iy$, $\exp(z) = \exp(x+iy)$

$= \exp(x)\cdot\underbrace{\exp(iy)}_{\cos(y)+i\sin(y)}$

Euler's formula

$$\boxed{\begin{aligned} |\exp(iy)|^2 &= \exp(iy)\exp(-iy) \\ &= \exp(iy-iy) \\ &= \exp(0) = 1 \end{aligned}}$$



define: $\frac{\pi}{2} :=$ smallest positive zero of $\cos: \mathbb{R} \to \mathbb{R}$

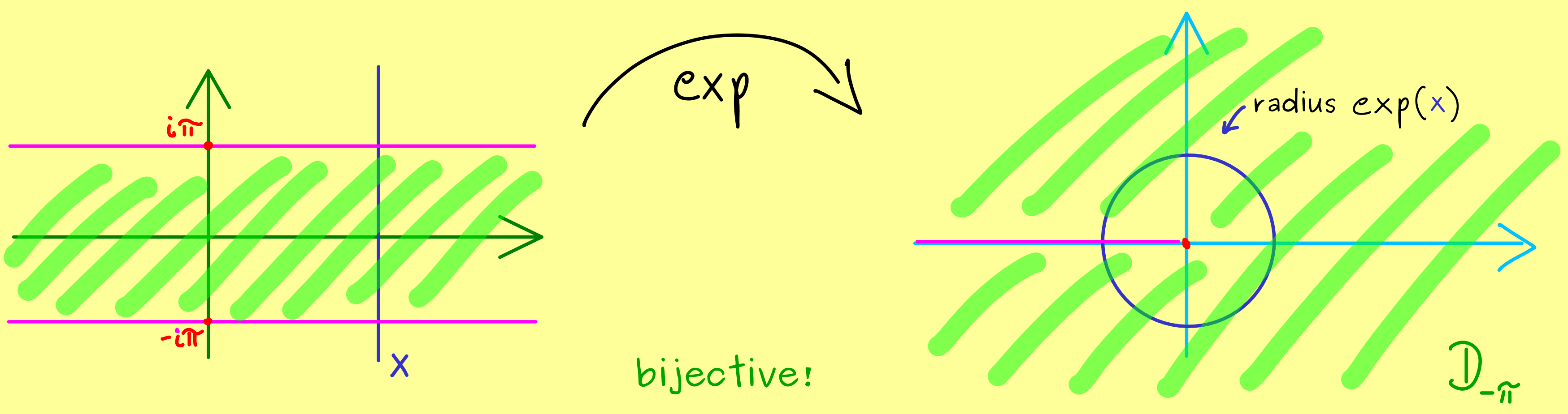
We get: $\exp\left(i\cdot\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$= i\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = i$ (use derivative/monotonicity)

$\exp(i\cdot\pi) = -1$ and $\exp(i\cdot 2\pi) = 1$

Periodicity: $\exp(z + 2\pi i\cdot k) = \exp(z)$ for all $k \in \mathbb{Z}$, $z \in \mathbb{C}$

↳ not injective



Definition: $\log: D_{-\pi} \longrightarrow$ stripe is called the principal value of the logarithm function.

Properties: $\log(r\exp(i\varphi)) = \log(r) + i\varphi$, $\varphi \in (-\pi, \pi)$

$\log(\exp(i\varphi)) \xrightarrow{\varphi \to \pi} i\pi$

$\log(\exp(i\varphi)) \xrightarrow{\varphi \to -\pi} -i\pi$

$2\pi i$ jump