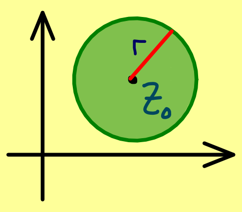
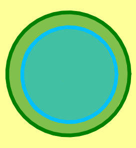
The Bright Side of Mathematics

Complex Analysis – Part 11

Result for power series: Let $f: B_r(z_0) \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$ be a power series with radius of convergence $r > 0$.



Then: (1) $\sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$



(2) $\sum_{k=1}^{\infty} a_k \cdot k (z - z_0)^{k-1}$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$

(3) f is complex differentiable with $f'(z) = \sum_{k=1}^{\infty} a_k \cdot k (z - z_0)^{k-1}$

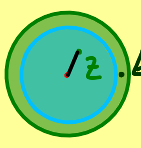
Proof: Assume $z_0 = 0$. $f_n: \overline{B_c(0)} \to \mathbb{C}$, $f_n(z) = \sum_{k=0}^{n} a_k \cdot z^k$

(1) $$\| f - f_n \|_\infty = \sup_{z \in \overline{B_c(0)}} \left| \sum_{k=n+1}^{\infty} a_k \cdot z^k \right| = \sup_{z \in \overline{B_c(0)}} \lim_{N \to \infty} \left| \sum_{k=n+1}^{N} a_k \cdot z^k \right|$$

supremum norm on $\overline{B_c(0)}$

Δ-inequality

$$\leq \sup_{z \in \overline{B_c(0)}} \lim_{N \to \infty} \sum_{k=n+1}^{N} |a_k| \cdot \underbrace{|z|^k}_{\leq c}$$



$\sum_{k=0}^{\infty} a_k \cdot \tilde{r}^k$ convergent for $c < \tilde{r} < r$

Hence there is B with $|a_k \tilde{r}^k| \leq B$

$B \geq |a_k| \cdot \tilde{r}^k = |a_k| \cdot c^k \cdot \Big(\underbrace{\frac{\tilde{r}}{c}}_{q^{-1}}\Big)^k$

$$\leq \sum_{k=n+1}^{\infty} |a_k| \cdot c^k \leq B \cdot \sum_{k=n+1}^{\infty} q^k \xrightarrow{n \to \infty} 0$$

(2) radius of convergence for $\sum_{k=1}^{\infty} \underbrace{a_k \cdot k}_{b_{k-1}} \cdot z^{k-1}$:

$$\limsup_{k \to \infty} \sqrt[k]{|a_{k+1}| \cdot (k+1)} = r$$

same proof as in (1)

(3) $\tilde{f}(z) := \sum_{k=1}^{\infty} a_k \cdot k \cdot z^{k-1}$, $p_N(z) := \sum_{k=0}^{N} a_k \cdot z^k$, $q_N(z) := \sum_{k=N+1}^{\infty} a_k \cdot z^k$

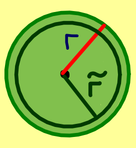
$$\left| \frac{f(z+h) - f(z)}{h} - \tilde{f}(z) \right| = \left| \frac{(p_N + q_N)(z+h) - (p_N + q_N)(z)}{h} - \tilde{f}(z) \right|$$

$$\leq \underbrace{\left| \frac{p_N(z+h) - p_N(z)}{h} - p_N'(z) \right|}_{A} + \underbrace{\left| p_N'(z) - \tilde{f}(z) \right|}_{B} + \underbrace{\left| \frac{q_N(z+h) - q_N(z)}{h} \right|}_{C}$$

$A \xrightarrow{h \to 0} 0$ $B \xrightarrow{N \to \infty} 0$ $C \xrightarrow{N \to \infty} 0$

For C: $$\left| \frac{\sum_{k=N+1}^{\infty} a_k (z+h)^k - \sum_{k=N+1}^{\infty} a_k \cdot z^k}{h} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \cdot \left| \underbrace{\frac{(z+h)^k - z^k}{h}} \right|$$

$$\frac{(z+h)^k - z^k}{h} = (z+h)^{k-1} \cdot z^0 + (z+h)^{k-2} \cdot z^1 + \dots$$
$$\dots + (z+h)^0 \cdot z^{k-1}$$



Geometric sum formula:

$$\frac{1 - q^k}{1 - q} = \sum_{j=0}^{k-1} q^j$$

Choose: $q = \frac{z}{z+h}$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \cdot \tilde{r}^{k-1} \cdot k \xrightarrow{N \to \infty} 0$$