Complex Analysis – Part 9

Power series

Example: Exponential function: $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Definition: For a sequence of complex numbers $a_0, a_1, a_2, a_3, \dots$,

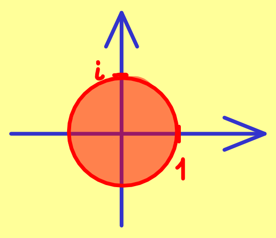
the function $f: D \to \mathbb{C}$, $z \mapsto \sum_{k=0}^{\infty} a_k (z - z_0)^k$ expansion point

with $D := \left\{ z \in \mathbb{C} \;\middle|\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ is convergent} \right\}$

is called a power series.

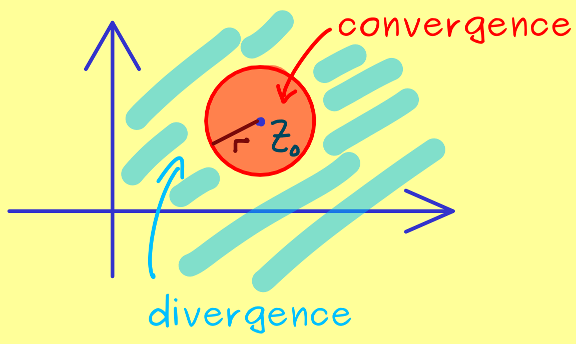
Example: Geometric series: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z| < 1$

$D = B_1(0)$ divergent for $|z| \geq 1$



Fact: For $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, there is a maximal $r \in [0, \infty) \cup \{\infty\}$

such that $\begin{cases} B_r(z_0) \subseteq D & \text{for } r \in [0,\infty) \\ \mathbb{C} = D & \text{for } r = \infty \end{cases}$



and for $z \in \mathbb{C} \setminus \overline{B_r(z_0)}$ the power series is divergent.

Cauchy–Hadamard: $\frac{1}{r} = \limsup_{k \to \infty} \sqrt[k]{|a_k|} \in [0, \infty) \cup \{\infty\}$ $\left(\frac{1}{0} = \infty, \; \frac{1}{\infty} = 0 \right)$

r is called the radius of convergence.