The Bright Side of Mathematics

Complex Analysis – Part 8

$f: U \to \mathbb{C}$ holomorphic

$\frac{\partial f}{\partial z}(z_0)$ Wirtinger derivatives $\frac{\partial f}{\partial \bar{z}}(z_0)$

$\frac{\partial f}{\partial z}(z_0) \overset{?}{=} f'(z_0)$, $\quad \frac{\partial f}{\partial \bar{z}}(z_0) \overset{?}{=} 0$

$$f'(x+iy) = \underbrace{a}_{\frac{\partial u}{\partial x}(x,y)} + i \underbrace{b}_{\frac{\partial v}{\partial x}(x,y)}$$

for $f_\mathbb{R}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

and map $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto f(x+iy) = u(x,y) + i\, v(x,y)$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial y}} + i \underbrace{\frac{\partial v}{\partial x}}_{-\frac{\partial u}{\partial y}} \right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{\partial}{\partial x}(u+iv)}_{\frac{\partial f}{\partial x}} - i \underbrace{\frac{\partial}{\partial y}(u+iv)}_{\frac{\partial f}{\partial y}} \right)$$

<u>Definition:</u> $\frac{\partial}{\partial z} := \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$

<u>Example:</u> $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i \cdot 2 \cdot x \cdot y \Rightarrow \frac{\partial f}{\partial x} = 2 \cdot x + i 2 y = 2 \cdot z$

$\frac{\partial f}{\partial y} = -2y + i 2x = 2 \cdot i\, z$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(2z + i \cdot 2 i z\right) = 0 \quad , \quad \frac{\partial f}{\partial z} = \frac{1}{2}\left(2z - i \cdot 2 i z\right) = 2 \cdot z$$

<u>Fact:</u> $f: U \to \mathbb{C}$ holomorphic $\Longleftrightarrow$ $\frac{\partial f}{\partial \bar{z}} = 0$ at all points in U

In this case: $f' = \frac{\partial f}{\partial z}$