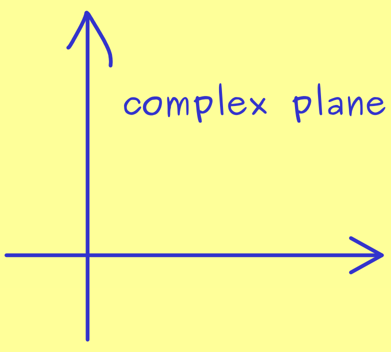
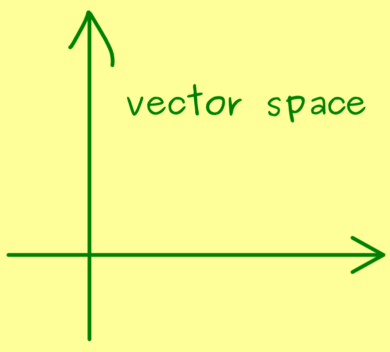
Complex Analysis – Part 5

$\mathbb{C}$:


$\mathbb{R}^2$:


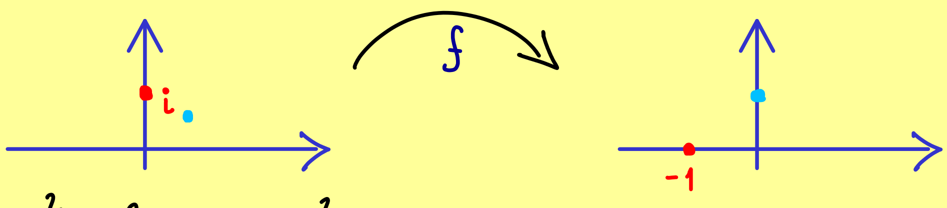
$\mathbb{C}$ is $\mathbb{R}^2$ with a multiplication

Remember: Each map $f : \mathbb{C} \to \mathbb{C}$ induces a map $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ (and vice versa)

Example: $f : \mathbb{C} \to \mathbb{C}$

$$z \mapsto z^2$$

$$x + iy \mapsto (x+iy)^2 = x^2 + 2ixy - y^2$$



$$f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

$$f_{\mathbb{R}}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Definition: A map $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if there is a matrix $J \in \mathbb{R}^{2\times 2}$ and a map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with:

$$f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underbrace{f_{\mathbb{R}}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)}_{\text{linear approximation}} + \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where $\dfrac{\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} 0$

$\sqrt{(x-x_0)^2 + (y-y_0)^2}$ = (Euclidean) norm

J is called the Jacobian matrix of $f_{\mathbb{R}}$ at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$.

$$J = \begin{pmatrix} | & | \\ \frac{\partial f_{\mathbb{R}}}{\partial x} & \frac{\partial f_{\mathbb{R}}}{\partial y} \\ | & | \end{pmatrix} \qquad \left(\text{evaluate at } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)$$

Example: $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \qquad J = \begin{pmatrix} 2\cdot x & -2\cdot y \\ 2\cdot y & 2\cdot x \end{pmatrix}$$